# STEP II 2007 Marking Scheme

**1** (i) $\left(1+\frac{k}{100}\right)^{\frac{1}{2}} = 1 + \left(\tfrac{1}{2}\right).\frac{k}{100} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2.1}.\frac{k^2}{10\,000} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3.2.1}.\frac{k^3}{1\,000\,000} + \ldots..$

**M1 for clear attempt at binomial expansion**

$$= 1 + \frac{k}{200} - \frac{k^2}{80\,000} + \frac{k^3}{16\,000\,000} + \ldots..$$

**A1 A1 A1 One each coefft.**

**OK if left in factorised/standard form or 3/48 not cancelled.**

**If left in 1st line form, give the A1 for the coefft. of *k* (others may be recouped later).** **4**

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(a) When $k = 8$, LHS $= \sqrt{1.08} = \sqrt{\frac{3\times 36}{100}} = \frac{6}{10}\sqrt{3}$

**M1 for extracting a $\sqrt{3}$ on the LHS** **A1 for correct multiple**

Then $\frac{6}{10}\sqrt{3} \approx 1 + \frac{8}{200} - \frac{64}{80\,000} + \frac{512}{16\,000\,000} + \ldots..$

**M1 for subst$^g$. $k = 8$ in RHS**

$= 1 + 0.04 - 0.0008 + 0.000\,032 + \ldots.. = 1.039\,232$ or equivalent **A1**

**M1 for converting fractions to decimals and attempt at the arithmetic (including $\times 10$ and $\div 6$ to get $\sqrt{3}$ only)**

$\Rightarrow \sqrt{3} \approx 1.732\,05$ (to 5 d.p.) **A1 for exactly this** **6**

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(b) **B1 for choosing** $k = -4$

**M1 for showing LHS a rational multiple of $\sqrt{6}$** LHS $= \sqrt{0.96} = \frac{4}{10}\sqrt{6}$

$\frac{4}{10}\sqrt{6} \approx 1 + -0.02 - 0.0002 - 0.000\,004 + \ldots.. = 0.979\,796$

**M1 for subst$^g$. into RHS and attempting the arithmetic (including $\times 10$ and $\div 4$ to get $\sqrt{6}$ only)**

$\Rightarrow \sqrt{6} \approx 2.449\,49$ (to 5 d.p.) **A1 for exactly this**

NB Choosing $k = 50$, gives $\frac{1}{2}\sqrt{6}$ **(B1 M1)** $= 1 + \frac{1}{4} - \frac{1}{32} + \frac{1}{128} + \ldots = 1.226\,562\,5$

Mult$^g$. by 2 **(M1)** gives $\sqrt{6} \approx 2.453\,125$ **(A0)** **4**

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(ii) $\left(1+\frac{k}{1000}\right)^{\frac{1}{3}} = 1 + \left(\tfrac{1}{3}\right).\frac{k}{1000}$ **M1 for clear attempt at binomial expansion** $= 1 + \frac{k}{3\,000}$ **A1 (*k* coefft.)**

**B1 for choosing** $k = 29$ **M1 for showing LHS a multiple of $\sqrt[3]{3}$** LHS $= \sqrt[3]{\frac{3\times 343}{1000}} = \frac{7}{10}\sqrt[3]{3}$

$\frac{7}{10}\sqrt[3]{3} \approx 1 + \frac{29}{3000} = \frac{3029}{3000}$

**M1 for subst$^g$. into RHS and attempting the arithmetic (including $\times 10$ and $\div 7$ to get $\sqrt[3]{3}$ only)**

$\Rightarrow \sqrt[3]{3} \approx \frac{3029}{2100}$ **A1 for exactly this (GIVEN ANSWER)** **6**

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**2** **(i)** $y = 2x^3 - bx^2 + cx \Rightarrow \dfrac{dy}{dx} = 6x^2 - 2bx + c$ **B1 for diff$^{g}$. correctly**

$\equiv 6(x - p)(x - q) \equiv 6x^2 - 6(p + q)x + 6pq$

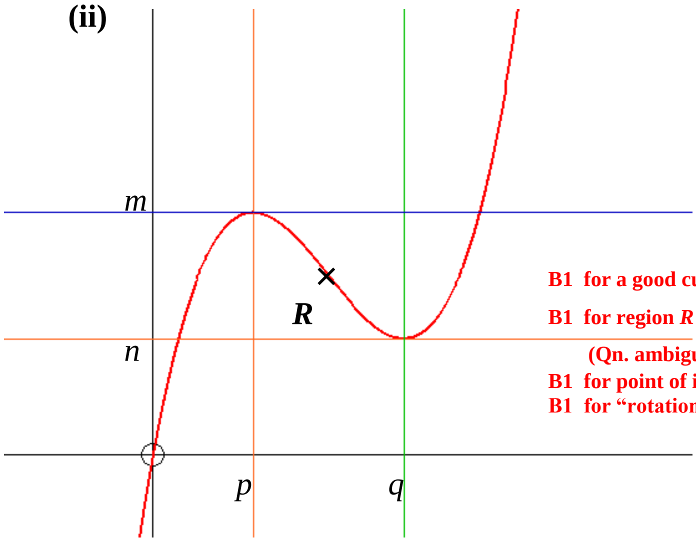
**M1 for equating to quadratic with roots *p* and *q***

$\Rightarrow b = 3(p + q)$ and $c = 6pq$

**A1** **A1**

**4**

**(ii)**




**B1 for a good cubic, with 2 TPs, through *O***

**B1 for region *R* correctly identified**

**(Qn. ambiguous – give for region to the left of this *R* also)**

**B1 for point of inflection indicated on curve**

**B1 for "rotational symmetry (of order 2) about P of I"**

**4**

**(iii)** $y = 2x^3 - 3(p + q)x^2 + 6pqx$ **M1 for subst$^{g}$. *p* and *q* into their eqn. for *y* (with *p*s and *q*s in it)**

$x = p \Rightarrow y = 2p^3 - 3(p + q)p^2 + 6p^2q \Rightarrow m = (3q - p)p^2$ **A1 for *m* correct**

$x = q \Rightarrow y = 2q^3 - 3(p + q)q^2 + 6pq^2 \Rightarrow n = (3p - q)q^2$ **A1 for *n* correct**

so that $m - n = q^3 - 3q^2p + 3qp^2 - p^3 = (q - p)^3$ **A1 cao (GIVEN ANSWER)**

**4**

**(iv) LONG METHOD**

Area under curve = $\displaystyle\int_p^q \left(2x^3 - 3(p+q)x^2 + 6pq\right)\,dx$ **M1 for attempt**

$= \left[\tfrac{1}{2}x^4 - (p+q)x^3 + 3pqx^2\right]_p^q$ **A1 for correct intgr$^{n}$. (just the "*y*" bit for now)**

$= \tfrac{1}{2}\left(q^4 - p^4\right) - (p+q)\left(q^3 - p^3\right) + 3pq\left(q^2 - p^2\right)$

**M1 for correct subst$^{n}$. of limits** **M1 for factorising answer (attempt)**

**A1 for correct answer** $= \tfrac{1}{2}(q - p)(q + p)\left\{4pq - q^2 - p^2\right\}$

**M1 for** Area required = Area under curve – $n(q - p)$

**M1 for using their *n* and attempting to tidy it all up**

$= \tfrac{1}{2}(q - p)\left\{4pq^2 + 4p^2q - q^3 - q^2p - qp^2 - p^3 - 2q^2(3p - q)\right\}$

$= \tfrac{1}{2}(q - p)^4$

**A1 for correct answer from completely correct working (GIVEN ANSWER)**

**(iv) SHORT METHOD**

Since the cubic is rotationally symmetric about its point-of-inflection,
Area required = $\frac{1}{2}$ × rectangle shown on above diagram **M5**

$= \frac{1}{2}(q-p)(m-n)$ **A1**

**M1 for using (iii)s result**

$= \frac{1}{2}(q-p)^4$ **A1 (GIVEN ANSWER)** **8**

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For candidates finding the LH possible $R$ …..

Using $n = q^2(3p - q) = 2x^3 - 3(p + q)x^2 + 6pqx$

$\Rightarrow\ 0 = 2x^3 - 3(p + q)x^2 + 6pqx - q^2(3p - q)$

and comparing this to

$0 = 2(x - q)^2\,(x - a) = 2x^3 - 2(a + 2q)x^2 + 2q(q + 2a)x - 2q^2a$

gives $a = \frac{1}{2}(3p - q)$ **M1 A1 for this extra work**

Then $A = \int_a^p \left(2x^3 - 3(p+q)x^2 + 6pq\right)\,\mathrm{d}x - n(p - a)$

Marks, as before, are
**M1 A1 for the integration of the "*y*" bit**
**M1 for correct use of limits throughout**
**M1 for correct difference of areas**
**M1 for using their *n* and attempting to tidy everything up**
**A1 for the correct final answer** $= \frac{11}{32}(q-p)^4$

Unfortunately, it is impossible to say whether such folks might have spotted the slick approach after choosing the "wrong" $R$.

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**3** $x = a\tan\theta \Rightarrow \mathrm{d}x = a\sec^2\theta\,\mathrm{d}\theta$

$\displaystyle\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \int \frac{1}{a^2\left(1+\tan^2\theta\right)}\,a\sec^2\theta\,\mathrm{d}\theta$ **M1 for full subst^n^.**

$\displaystyle = \int \frac{1}{a}\,\mathrm{d}\theta = \frac{1}{a}.\theta = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + \text{constant}$ **A1 (GIVEN ANSWER)** **2**

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**(i) (a)** **M1 for use of $s = \sin x$ subst^n^. (or equivalent)** $s = \sin x \Rightarrow \mathrm{d}s = \cos x\,\mathrm{d}x$ and $\left(0, \tfrac{1}{2}\pi\right) \to (0, 1)$

**A1 for** $\displaystyle I = \int_0^1 \frac{\mathrm{d}s}{1+s^2}$ **(ignore incorrect limits here)** $= \left[\tan^{-1} s\right]_0^1$ **A1** $\displaystyle = \frac{\pi}{4}$ **A1 cao** **4**

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**(b) FORWARDS APPROACH**

$t = \tan\tfrac{1}{2}x \Rightarrow \mathrm{d}t = \tfrac{1}{2}\sec^2\tfrac{1}{2}x\,\mathrm{d}x \Rightarrow \mathrm{d}x = \dfrac{2\,\mathrm{d}t}{1+t^2}$ **B1** **B1 for correct limits (0, 1)**

Then $\displaystyle I = \int_0^1 \left(\frac{\left(\dfrac{1-t^2}{1+t^2}\right)}{\left(\dfrac{1+t^2}{1+t^2}\right)^2 + \dfrac{4t^2}{\left(1+t^2\right)^2}}\right) \times \frac{2\,\mathrm{d}t}{1+t^2}$ $\displaystyle = 2\int_0^1 \frac{\left(1-t^2\right)}{1+6t^2+t^4}\,\mathrm{d}t$

**M1 for full subst^n^.**
**B1 B1 – one for each of $\cos x$ and $\sin x$ correctly in terms of $t$** **A1 (GIVEN ANSWER)**

**(b) BACKWARDS APPROACH**

$t = \tan\tfrac{1}{2}x \Rightarrow \mathrm{d}t = \tfrac{1}{2}\sec^2\tfrac{1}{2}x\,\mathrm{d}x \Rightarrow \mathrm{d}x = \dfrac{2\,\mathrm{d}t}{1+t^2}$ **B1** **B1 for correct limits (0, ½π)**

$\displaystyle\int_0^{\pi/2} \frac{1-\tan^2\tfrac{1}{2}x}{1+6\tan^2\tfrac{1}{2}x+\tan^4\tfrac{1}{2}x} \times \tfrac{1}{2}\left(1+\tan^2\tfrac{1}{2}x\right)\,\mathrm{d}x$ **M1 for full subst^n^.**

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1-\tan^4\tfrac{1}{2}x}{1+6\tan^2\tfrac{1}{2}x+\tan^4\tfrac{1}{2}x}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\cos^4\tfrac{1}{2}x-\sin^4\tfrac{1}{2}x}{\cos^4\tfrac{1}{2}x+6\sin^2\tfrac{1}{2}x\cos^2\tfrac{1}{2}x+\sin^4\tfrac{1}{2}x}\,\mathrm{d}x$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\left(c^2-s^2\right)\left(c^2+s^2\right)}{\left(c^2+s^2\right)^2-2s^2c^2+6s^2c^2}\,\mathrm{d}x = \frac{1}{2}\int_0^{\pi/2} \frac{\cos x \times 1}{1+\sin^2 x}\,\mathrm{d}x$$

**B1 B1 – one for each of $\cos x$ and $\sin x$ correctly in terms of $t$** **A1 (GIVEN ANSWER)** **6**

**(ii)** **M1 for working backwards on** $\displaystyle\int_0^1 \frac{1-t^2}{1+14t^2+t^4}\,\mathrm{d}t$ **to get something of the form**

$$\tfrac{1}{2}\int_0^1 \left( \frac{\left(\dfrac{1-t^2}{1+t^2}\right)}{\left(\dfrac{1+t^2}{1+t^2}\right)^2 + 3\left(\dfrac{2t}{1+t^2}\right)^2} \right) \times \frac{2\,\mathrm{d}t}{1+t^2}$$ **A1 for correct unsimplified**

**A1 for** $\displaystyle\tfrac{1}{2}\int_0^{\pi/2} \frac{\cos x}{1+3\sin^2 x}\,\mathrm{d}x$

**M1 for subst$^{n}$.** $s = \sqrt{3}\,\sin x$ (or equivalent)

**B1 for** $\mathrm{d}s = \sqrt{3}\,\cos x$ and $(0,\ \tfrac{1}{2}\pi) \to (0,\ \sqrt{3})$

**A1 for** $\displaystyle I = \frac{1}{2\sqrt{3}}\int_0^{\sqrt{3}} \frac{\mathrm{d}s}{1+s^2}$ **(ignore limits here)**

$$= \frac{1}{2\sqrt{3}}\left[\tan^{-1} s\right]_0^{\sqrt{3}}$$ **A1**

$$= \frac{\pi}{6\sqrt{3}}$$ **A1 (or exact equivalent)**

**8**

**4** **M1 for use of sin($A$ – $B$), cos($A$ + $B$) and sin($A$ + $B$) formulae**

$\alpha[\sin A\cos B - \cos A\sin B] + \beta[\cos A\cos B - \sin A\sin B] = \gamma[\sin A\cos B + \cos A\sin B]$

**A1 all correct unsimplified**

**M1 for dividing by cos $A$ cos $B$ throughout**

**A1 for** $\alpha\tan A - \alpha\tan B + \beta - \beta\tan A\tan B = \gamma\tan A + \gamma\tan B$

**M1 for collecting up into the form** $0 = \tan A\tan B + \left(\dfrac{\gamma-\alpha}{\beta}\right)\tan A + \left(\dfrac{\gamma+\alpha}{\beta}\right)\tan B - 1$

**M1 for the following factorisation attempt** $0 = \left[\tan A + \left(\dfrac{\gamma+\alpha}{\beta}\right)\right]\left[\tan B - \left(\dfrac{\alpha-\gamma}{\beta}\right)\right] + \dfrac{\alpha^2-\gamma^2}{\beta^2} - 1$

**For noting this is of the form** $0 = (\tan A - m)(\tan B - n)$ where

$m = -\left(\dfrac{\gamma+\alpha}{\beta}\right)$, $n = \left(\dfrac{\alpha-\gamma}{\beta}\right)$ and $\dfrac{\alpha^2-\gamma^2}{\beta^2} = 1$ i.e. $\alpha^2 = \beta^2 + \gamma^2$

**A1 for both $m$, $n$** **A1 for this part**

**Ignore lack of ⇔ established.** **8**

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**(i)** $\alpha = 2$, $\beta = \sqrt{3}$ and $\gamma = 1$ satisy the condition $\alpha^2 = \beta^2 + \gamma^2$ **B1 noted**

**M1 for** $0 = \left[\tan x + \sqrt{3}\right]\left[\tan\dfrac{\pi}{4} - \dfrac{1}{\sqrt{3}}\right] \Rightarrow \tan x = -\sqrt{3}$

**A1 for both answers** $x = \dfrac{2\pi}{3}$ and $\dfrac{5\pi}{3}$ **3**

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**(ii)** **M1 for** $0 = \left[\tan x + \sqrt{3}\right]\left[\tan\dfrac{\pi}{6} - \dfrac{1}{\sqrt{3}}\right]$

**B1 for noting 2nd bracket ≡ zero** **A1 for observing that** statement is true for all $x \in [0, 2\pi)$ **3**

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**(iii)** **M1 for noting** $A = 2x + \dfrac{\pi}{6}$ and $B = x - \dfrac{\pi}{6}$ so that $0 = \left[\tan x + \sqrt{3}\right]\left[\tan B - \dfrac{1}{\sqrt{3}}\right]$ **A1**

**M1 for** $\tan\left[2x + \dfrac{\pi}{6}\right] = \tan\dfrac{2\pi}{3}, \tan\dfrac{5\pi}{3}, \tan\dfrac{8\pi}{3}, \tan\dfrac{11\pi}{3}, \ldots$

**A1 for all answers** $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$ **and no extras** (ignore correct out-of-range answers)

**M1 for** $\tan\left[x - \dfrac{\pi}{6}\right] = \tan\dfrac{\pi}{6}, \tan\dfrac{7\pi}{6}, \ldots$

**A1 for all answers** $x = \dfrac{\pi}{3}, \dfrac{4\pi}{3}$ **and no extras (ignore correct out-of-range answers)** **6**

**ALTERNATIVE to (iii)**

$2\ \sin\left(x+\frac{\pi}{3}\right)$ = sin 3$x$ – $\sqrt{3}$ cos 3$x$ = $2\sin\left(3x-\frac{\pi}{3}\right)$ **M1 A1**

giving

$x+\frac{\pi}{3}=3x-\frac{\pi}{3}$ , $3x-\frac{\pi}{3}-2\pi$ , $3x-\frac{\pi}{3}-4\pi$ , ….. **M1 for general approach (either case)**

or

$x+\frac{\pi}{3}=\pi-\left(3x-\frac{\pi}{3}\right)$ , $3\pi-\left(3x-\frac{\pi}{3}\right)$ , $5\pi-\left(3x-\frac{\pi}{3}\right)$ , …..

i.e.

$x = \frac{\pi}{3}$ , $\frac{4\pi}{3}$ , ….. or $x = \frac{\pi}{4}$ , $\frac{3\pi}{4}$ , $\frac{5\pi}{4}$ , $\frac{7\pi}{4}$ , …..

**A1 for any 2 correct** **+ A1 for further two correct**

**+ A1 for all 6 with no extras – ignore correct answers outside [0, 2π)** **6**

**5** (i) $\mathrm{f}\{\,\mathrm{f}(x)\,\} = \dfrac{\left(\dfrac{x+\sqrt{3}}{1-\sqrt{3}\,x}\right)+\sqrt{3}}{1-\sqrt{3}\left(\dfrac{x+\sqrt{3}}{1-\sqrt{3}\,x}\right)}$ **M1 for genuine attempt at $\mathrm{f}^2$**

$= \dfrac{x+\sqrt{3}+\sqrt{3}-3x}{1-\sqrt{3}\,x-\sqrt{3}\,x-3}$ **dM1 for simplifying fraction** $= \dfrac{x-\sqrt{3}}{1+\sqrt{3}\,x}$ **A1**

**M1 for** $\mathrm{f}^3(x) = \mathrm{f}^2\{\,\mathrm{f}(x)\,\}$ or $\mathrm{f}\{\,\mathrm{f}^2(x)\,\}$ **genuinely attempted (ft their $\mathrm{f}^2$)**

$$= \frac{\left(\dfrac{x+\sqrt{3}}{1-\sqrt{3}\,x}\right)-\sqrt{3}}{1+\sqrt{3}\left(\dfrac{x+\sqrt{3}}{1-\sqrt{3}\,x}\right)} \quad\text{or}\quad \frac{\left(\dfrac{x-\sqrt{3}}{1+\sqrt{3}\,x}\right)+\sqrt{3}}{1-\sqrt{3}\left(\dfrac{x-\sqrt{3}}{1+\sqrt{3}\,x}\right)}$$

$$= \frac{x+\sqrt{3}-\sqrt{3}+3x}{1-\sqrt{3}\,x+\sqrt{3}\,x+3} \quad\text{or}\quad \frac{x-\sqrt{3}+\sqrt{3}+3x}{1+\sqrt{3}\,x-\sqrt{3}\,x+3} = x$$ **A1 cao**

Since $\mathrm{f}^3 = x$, the sequence $\{\mathrm{f}, \mathrm{f}^2, \mathrm{f}^3, \ldots\}$ is periodic with period 3.
**B1 for explanation**

As $2007 = 3 \times 669$ (or just is a multiple of 3), $\mathrm{f}^{2007}(x) = x$. **B1 for answer** **7**

(ii) $x = \tan\theta \Rightarrow \mathrm{f}(x) = \dfrac{\tan\theta+\tan\frac{\pi}{3}}{1-\tan\theta\tan\frac{\pi}{3}} = \tan(\theta+\frac{\pi}{3})$ **M1 A1 for showing true for $n = 1$**

**EITHER**
a semi-formal inductive proof:
i.e. assume that $\mathrm{f}^k(x) = \tan(\theta + \frac{1}{3}k\pi)$ for some integer $k \geq 1$ **B1**

then $\mathrm{f}^{k+1}(x) = \mathrm{f}\{\,\mathrm{f}^k(x)\,\} = \dfrac{\tan[\theta+k\frac{\pi}{3}]+\sqrt{3}}{1-\sqrt{3}\tan[\theta+k\frac{\pi}{3}]}$ **M1 for attempt at inductive step**

$= \dfrac{\tan[\theta+k\frac{\pi}{3}]+\tan\frac{\pi}{3}}{1-\tan[\theta+k\frac{\pi}{3}]\tan\frac{\pi}{3}}$ **dM1 for correct use of fn. composition with $\mathrm{f}^k$**

$= \tan[\theta+(k+1)\frac{\pi}{3}]$ **A1 for fully correct proof**

**OR**

$\mathrm{f}^2(x) = \dfrac{\tan\theta-\sqrt{3}}{1+\sqrt{3}\tan\theta} = \dfrac{\tan\theta+\tan\frac{2\pi}{3}}{1-\tan\theta\tan\frac{2\pi}{3}} = \tan(\theta+\frac{2\pi}{3})$ **M1 A1**

and $\mathrm{f}^3(x) = \tan\theta = \tan(\theta+\frac{3\pi}{3})$ **B1**

and observation that $\tan(A+\pi) = \tan A$ now gives

$\mathrm{f}^n(x) = \tan(\theta+\frac{1}{3}n\pi)$ $\forall$ positive integers $n$ **B1** **6**

(iii) Let $t = \sin\theta$ **M1** Then $\sqrt{1-t^2} = \cos\theta$ **A1**

and

$g(t) = \sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6} = \sin(\theta + \frac{\pi}{6})$ or $\sin(\sin^{-1}t + \frac{\pi}{6})$ **B1**

$g^n(t) = \sin(\sin^{-1}t + \frac{n\pi}{6})$ **B1 for identifying correct result**

Assuming $g^k(t) = \sin(\sin^{-1}t + \frac{k\pi}{6})$ **M1 for induction attempt**

$$g^{k+1}(t) = \sin\left[\sin^{-1}\{\sin(\sin^{-1}t + \tfrac{k\pi}{6})\} + \tfrac{\pi}{6}\right] \quad \textbf{dM1}$$

$$= \sin\left[\sin^{-1}t + \tfrac{k\pi}{6} + \tfrac{\pi}{6}\right]$$

$$= \sin\left[\sin^{-1}t + \tfrac{(k+1)\pi}{6}\right] \quad \textbf{A1}$$

Hence, by induction, $g^n(t) = \sin(\sin^{-1}t + \frac{n\pi}{6})$ for all positive integers $n$ **7**

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**ALTERNATIVE to (iii)**

For $t = \sin\theta$ **M1** $\sqrt{1-t^2} = \cos\theta$ **A1**

and

$g(t) = \sin\theta\sin\frac{\pi}{3} + \cos\theta\cos\frac{\pi}{3} = \cos(\theta - \frac{\pi}{3})$ **B1**

But then $g^2(t) = \sin\theta = t$ **M1 A1**
and the sequence of powers of g is periodic, period 2 (which makes its description rather easy). **M1 A1**

The issue here is that, strictly speaking, $\sqrt{1-t^2} = |\cos\theta|$, and so different parts of the domain of g actually give different composite functions.

In fact, the function g is even weirder than that, but a deeper discussion is not appropriate here. Candidates should receive full marks for either approach correctly implemented, but gain no extra credit for spotting the dichotomy.

**6** (i) $y = \ln\left(x+\sqrt{3+x^2}\right) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x+\sqrt{3+x^2}} \times \left(1 + \dfrac{1}{2}\left(3+x^2\right)^{-\frac{1}{2}}.2x\right)$ **M1 for use of Chain Rule**

**A1 correct unsimplified**

**dM1 for simplifying attempt** $= \dfrac{1}{x+\sqrt{3+x^2}} \times \left(\dfrac{\sqrt{3+x^2}+x}{\sqrt{3+x^2}}\right)$

$= \dfrac{1}{\sqrt{3+x^2}}$ **A1 for correct answer**

$y = x\sqrt{3+x^2} \Rightarrow \dfrac{dy}{dx} = x.\dfrac{1}{2}\left(3+x^2\right)^{-\frac{1}{2}}.2x + \sqrt{3+x^2}$ **M1 for use of Product Rule**

**A1 correct unsimplified**

$= \dfrac{2x^2+3}{\sqrt{3+x^2}}$ **A1 for correct, simplified answer** **7**

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$\sqrt{3+x^2} = \dfrac{3+x^2}{\sqrt{3+x^2}} = \dfrac{1}{2} \times \dfrac{2x^2+3}{\sqrt{3+x^2}} + \dfrac{3}{2} \times \dfrac{1}{\sqrt{3+x^2}}$

**M1 for attempting to write as a linear sum of previous two answers (MUST be a "Hence" approach)**
**A1** **A1 one for each coefft.**

Hence $\int \sqrt{3+x^2}\,dx = \dfrac{1}{2}x\sqrt{3+x^2} + \dfrac{3}{2}\ln\left(x+\sqrt{3+x^2}\right)$ $(+\,C)$ **A1** **4**

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(ii) **M1 for treating** $3\left(\dfrac{dy}{dx}\right)^2 + 2x\dfrac{dy}{dx} = 1$ **as a quadratic in** $\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{1}{3}\left(\sqrt{3+x^2}-x\right)$ or $-\dfrac{1}{3}\left(\sqrt{3+x^2}+x\right)$ **dM1 for quadratic formula/compl[g]. the square**

**A1**

**M1 for good integration attempt at either (using previous results):**

$y = \dfrac{1}{6}x\sqrt{3+x^2} + \dfrac{1}{2}\ln\left(x+\sqrt{3+x^2}\right) - \dfrac{1}{6}x^2 + C_1$ **A1**

**These can be ft A marks from *constant* errors.**

$y = -\dfrac{1}{6}x\sqrt{3+x^2} - \dfrac{1}{2}\ln\left(x+\sqrt{3+x^2}\right) - \dfrac{1}{6}x^2 + C_2$ **A1**

**M1 for subst[g]. $x = 1,\ y = 0$ to find the "+ $C$"s**

$y = \dfrac{1}{6}x\sqrt{3+x^2} + \dfrac{1}{2}\ln\left(x+\sqrt{3+x^2}\right) - \dfrac{1}{6}x^2 - \dfrac{1}{6} - \dfrac{1}{2}\ln 3$ **A1 cao**

$y = -\dfrac{1}{6}x\sqrt{3+x^2} - \dfrac{1}{2}\ln\left(x+\sqrt{3+x^2}\right) - \dfrac{1}{6}x^2 + \dfrac{1}{2} + \dfrac{1}{2}\ln 3$ **A1 cao** **9**

**7** $f(x) = \sin x \Rightarrow f''(x) = -\sin x < 0$ on $(0, \pi)$ **M1 A1**

$f(x) = \ln x \Rightarrow f''(x) = -\frac{1}{x^2} < 0$ on $(0, \infty)$ **M1 A1** **4**

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(i) Using $\sin x$ on $(0, \pi)$ **M1**

$$\frac{1}{3}(\sin A + \sin B + \sin C) \le \sin\left(\frac{A+B+C}{3}\right)$$ **A1 to this stage**

Noting that $A + B + C = \pi$ **M1**

$$\Rightarrow \sin A + \sin B + \sin C \le 3\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}$$ **A1 (GIVEN ANSWER)** **4**

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(ii) Using $\ln x$ on $(0, \infty)$ **M1**

$$\frac{1}{n}(\ln t_1 + \ln t_2 + \ldots\ldots + \ln t_n) \le \ln\left(\frac{t_1 + t_2 + \ldots\ldots + t_n}{n}\right)$$ **A1 to this stage**

**M1 for use of log. laws to sort LHS appropriately**

$$\ln(t_1 \times t_2 \times \ldots\ldots \times t_n)^{\frac{1}{n}} \le \ln\left(\frac{t_1 + t_2 + \ldots\ldots + t_n}{n}\right)$$

$$\Rightarrow \sqrt[n]{t_1 t_2 \ldots\ldots t_n} \le \frac{t_1 + t_2 + \ldots\ldots + t_n}{n}$$ **A1 (GIVEN ANSWER)** **4**

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(a) Using the previous result (*the AM – GM inequality*) **M1 for attempt**

$$\frac{x^4 + y^4 + z^4 + 2^4}{4} \ge \sqrt[4]{x^4 . y^4 . z^4 . 2^4}$$ **A1 correct unsimplified**

i.e. $x^4 + y^4 + z^4 + 16 \ge 8xyz$ **A1** **3**

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(b) Use of *the AM – GM inequality* **M1 for attempt**

$$\frac{x^5 + y^5 + z^5 + 1^5 + 1^5}{5} \ge \sqrt[5]{x^5 . y^5 . z^5 . 1^5 . 1^5}$$ **A1 correct unsimplified**

i.e. $x^5 + y^5 + z^5 + 2 \ge 5xyz$ **A1** $\Rightarrow x^5 + y^5 + z^5 - 5xyz \ge -2$ **A1**

**B1 for crucial observation that** minimum actually occurs when $x = y = z = 1$ **5**

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|---|---|---|---|
| **8** | (i) | When $s + t = 1$, locus of $X$ is a straight line **M1** through $B$ and $C$ **A1** | **2** |
| | (ii) | Eqn. $AP$ is $\underline{\mathbf{r}} = \lambda(\beta\,\underline{\mathbf{b}} + \gamma\,\underline{\mathbf{c}})$ **B1**<br>Eqn. $BC$ is $\underline{\mathbf{r}} = \underline{\mathbf{b}} + \mu(\underline{\mathbf{c}} - \underline{\mathbf{b}})$ or $\underline{\mathbf{r}} = (1 - \mu)\,\underline{\mathbf{b}} + \mu\,\underline{\mathbf{c}}$ **B1**<br>Lines meet at $D$ when $\lambda\beta = 1 - \mu$ and $\lambda\gamma = \mu$ **M1**<br>**dM1 for solving:** $\lambda = \dfrac{1}{\beta+\gamma}$ **A1** and $\mu = \dfrac{\gamma}{\beta+\gamma}$ or $1 - \mu = \dfrac{\beta}{\beta+\gamma}$ **A1**<br>so that $D$ cuts $BC$ in the ratio $\mu : 1 - \mu = \gamma : \beta$ **M1 A1** | **8** |
| | (iii) | Eqn. $BP$ is $\underline{\mathbf{r}} = \underline{\mathbf{b}} + \lambda\,(\underline{\mathbf{p}} - \underline{\mathbf{b}})$ or $\underline{\mathbf{r}} = (1 - \lambda + \lambda\beta)\,\underline{\mathbf{b}} + \lambda\gamma\,\underline{\mathbf{c}}$ **B1**<br>Eqn. $AC$ is $\underline{\mathbf{r}} = \mu\,\underline{\mathbf{c}}$ **B1**<br>Lines meet at $E$ when $1 - \lambda + \lambda\beta = 0$ and $\lambda\gamma = \mu$ **M1 incl. attempt to solve for $\lambda$ and $\mu$**<br>$\lambda = \dfrac{1}{1-\beta}$ and $\mu = \dfrac{\gamma}{1-\beta}$ or $1 - \mu = \dfrac{1-\beta-\gamma}{1-\beta}$<br>so that $E$ cuts $AC$ in the ratio $1 - \mu : \mu = 1 - \beta - \gamma : \gamma$ **A1** | **4** |
| | | Eqn. $CP$ is $\underline{\mathbf{r}} = \underline{\mathbf{c}} + \lambda\,(\underline{\mathbf{p}} - \underline{\mathbf{c}})$ or $\underline{\mathbf{r}} = (1 - \lambda + \lambda\gamma)\,\underline{\mathbf{c}} + \lambda\beta\,\underline{\mathbf{b}}$ **B1**<br>Eqn. $AB$ is $\underline{\mathbf{r}} = \mu\,\underline{\mathbf{b}}$ **B1**<br>Lines meet at $F$ when $1 - \lambda + \lambda\gamma = 0$ and $\lambda\beta = \mu$ **M1 incl. attempt to solve for $\lambda$ and $\mu$**<br>$\lambda = \dfrac{1}{1-\gamma}$ and $\mu = \dfrac{\beta}{1-\gamma}$ or $1 - \mu = \dfrac{1-\beta-\gamma}{1-\gamma}$<br>so that $F$ cuts $AB$ in the ratio $\mu : 1 - \mu = \beta : 1 - \beta - \gamma$ **A1** | **4** |
| | | Then $\dfrac{AF}{FB} \times \dfrac{BD}{DC} \times \dfrac{CE}{EA} = \dfrac{\beta}{1-\beta-\gamma} \times \dfrac{\gamma}{\beta} \times \dfrac{1-\beta-\gamma}{\gamma} = 1$ **M1 A1** | **2** |